Stable Flocking of Mobile Agents, Part I: Fixed Topology

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Abstract

This is the first of a two-part paper that investigates the stability properties of a system of multiple mobile agents with double integrator dynamics. In this first part we generate stable flocking motion for the group using a coordination control scheme which gives rise to smooth control laws for the agents. These control laws are a combination of attractive/repulsive and alignment forces, ensuring collision avoidance and cohesion of the group and an aggregate motion along a common heading direction. In this control scheme the topology of the control interconnections is fixed and time invariant. The control policy ensures that all agents eventually align with each other and have a common heading direction while at the same time avoid collisions and group into a tight formation.

I. Introduction

Over the last years, the problem of coordinating the motion of multiple autonomous agents, has attracted significant attention. Besides the links of this issue to problems in biology, social behavior, statistical physics, and computer graphics, to name a few, research was partly motivated by recent advances in communication and computation. Considerable effort has been directed in trying to understand how a group of autonomous moving creatures such as flocks of birds, schools of fish, crowds of people [30], [17], or man-made mobile autonomous agents, can cluster in formations without centralized coordination.

Similar problems have been studied in ecology and theoretical biology, in the context of animal aggregation and social cohesion in animal groups [1], [20], [32], [9], [6]. A computer model mimicking animal aggregation was proposed in [23]. Following the work in [23] several other computer models have appeared in the literature (cf. [10] and the references therein), and led to creation of a new area in computer graphics known as *artificial life* [23], [27]. At the same time, several researchers in the area of statistical physics and complexity theory have addressed flocking and schooling behavior in the context of non-equilibrium phenomena in many-degree-of-freedom dynamical systems and self organization in systems of self-propelled particles [31], [29], [28], [18], [14], [25], [4], [12]. Similar problems have become a major thrust in systems and control theory, in the context of cooperative control, distributed control of multiple vehicles and formation control; see for example [16], [13], [21], [22], [5], [15], [7], [26], [11], [19].

The animal aggregation model of [23] aimed at generating computer animation of the motion of bird flocks and fish schools. It was based on three dimensional computational geometry of the sort normally used in computer animation or computer aided design. The author called the generic simulated flocking creatures “boids”. The basic flocking model consists of three simple steering behaviors which describe how an individual agent maneuvers based on the positions and velocities its nearby flockmates:

- **Separation:** steer to avoid crowding local flockmates.
- **Alignment:** steer towards the average heading of local flockmates.

- **Cohesion:** steer to move toward the average position of local flockmates.

Each mobile agent has direct access to the whole scene's geometric description. The superposition of these three rules results in all agents moving in a formation, with a common heading while avoiding collisions.

Generalizations of this model include a leader follower strategy, in which one agent acted as a group leader and the other agents would just follow the aforementioned cohesion/separation/alignment rules, resulting in leader following. Vicsek *et al.* [31] proposed such a model in 1995. Although developed independently, Vicsek's model turns out to be a special case of [23], in which all agents move with the same speed (no dynamics), and only follow an alignment rule. In [31], each agent heading is updated as the average of the headings of agent itself with its nearest neighbors plus some additive noise. Numerical simulations in [31] indicate a coherent collective motion, in which the headings of all agents converge to a common value. This was quite a surprising result in the physics community and was followed by a series of papers [3], [29], [28], [24], [18]. A proof of convergence for Vicsek's model (in the noise-free case) was given in [11].

The goal of this paper is to develop a mathematical model for the flocking phenomenon in [23], introduce the control laws that give rise to such behavior and provide a system theoretic justification by combining results from classical control theory, mechanics and algebraic graph theory. In this first part of the paper, we consider the case where the topology of the control interactions between the agents is fixed. Each agent regulates its position and orientation based on a fixed set of "neighbors". In this case, the control inputs for the agent are smooth. The case where the set of neighbors may change in time, depending on the relative distances between the agent and its flockmates, is treated separately in Part II. Here we show that under fixed control interconnection topology, the system of mobile agents is capable of coordinating itself so that all agents attain a common heading, they cluster to a tight formation and move in a collision free fashion. The cohesion and separation rules can be *decoupled* from alignment. The results of this paper are in correspondence with those of [11], where nearest neighbor alignment laws were applied to kinematic models of mobile agents. Specifically, it is shown that the addition of dynamics and cohesion/separation control actions do not affect the stability of the flocking motion.

This paper is organized as follows: in section II we define the problem addressed in this paper and sketch the solution approach. In section III we give a brief introduction on algebraic graph theory. The purpose of section IV is to introduce the distributed control scheme that will establish the desired flocking behavior, and analyze the stability of the closed loop system. The results of section IV are verified in section V via numerical simulations. Section VI summarizes and highlights new research directions.

II. Problem Description

Consider N agents, moving on the plane with dynamics described by:

$$\dot{r}_i = v_i \tag{1a}$$

$$\dot{v}_i = a_i \quad i = 1, \ldots, N\,, \tag{1b}$$

where $r_i = (x_i, y_i)^T$ is the position vector of agent i, $v_i = (\dot{x}_i, \dot{y}_i)^T$ is its velocity vector and $a_i = (u_x, u_y)^T$ its control (acceleration) input. The heading of agent i, θ_i, is defined as:

$$\theta_i = \arctan(\dot{y}_i, \dot{x}_i)\,. \tag{2}$$

Relative position vector between agents i and j is denoted $r_{ij} = r_i - r_j$.

Each agent is controlled through acceleration inputs. The objective is to have group coordination under a distributed control scheme. The control input for agent i is given as the combination of two control forces (Figure 1):

$$a_i = a_{r_i} + a_{\theta_i} \ . \tag{3}$$

The first component, a_{r_i}, is derived from the field produced by an artificial potential function, V_i, that depends on the relative distances between agent i and its flockmates. This term is responsible for collision avoidance and cohesion in the group. The second component, a_{θ_i} regulates the heading of agent i to the weighted average of that of its flockmates, acting in a direction normal to the velocity vector of i. These components do not contribute to the kinetic energy of the system and can be regarded as *gyroscopic forces* [2]. We will show that the combination of these control forces establish a stable a stable, collision free flocking motion for the group.

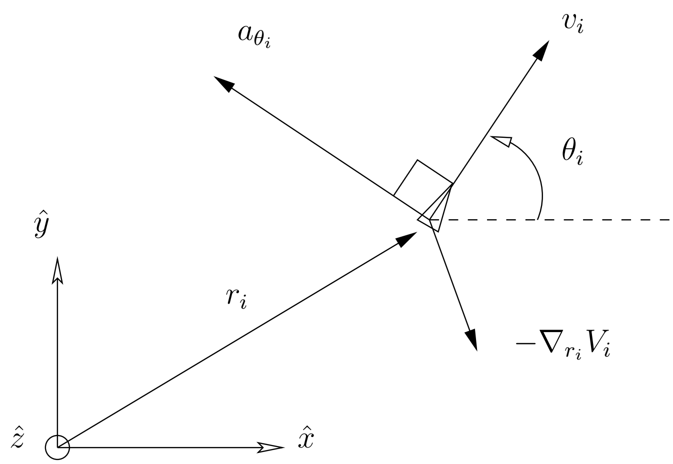

Fig. 1. Control forces acting on agent i.

III. Graph Theory Preliminaries

In the discussion that follows, we will use some graph theory terminology which we briefly introduce in this section. For more information, the interested reader is referred to [8].

An (undirected) graph $\mathcal{G}$ consists of a vertex set, $\mathcal{V}$, and an edge set $\mathcal{E}$, where an edge is an unordered pair of distinct vertices in $\mathcal{G}$. If $x, y \in \mathcal{V}$, and $(x, y) \in \mathcal{E}$, then x and y are said to be adjacent, or neighbors and we denote this by writing $x \sim y$. The number of neighbors of each vertex is its valency. A path of length r from vertex x to vertex y is a sequence of $r + 1$ distinct vertices starting with x and ending with y such that consecutive vertices are adjacent. If there is a path between any two vertices of a graph $\mathcal{G}$, then $\mathcal{G}$ is said to be connected.

The adjacency matrix $\mathcal{A}(\mathcal{G}) = [a_{ij}]$ of an (undirected) graph $\mathcal{G}$ is a symmetric matrix with rows and columns indexed by the vertices of $\mathcal{G}$, such that $a_{ij} = 1$ if vertex i and vertex j are neighbors and $a_{ij} = 0$, otherwise. The valency matrix $\Delta(\mathcal{G})$ of a graph $\mathcal{G}$ is a diagonal matrix with rows and columns indexed by $\mathcal{V}$, in which the (i, i)-entry is the valency of vertex i. An orientation of a graph $\mathcal{G}$ is the assignment of a direction to each edge, so that the edge (i, j) is now an arc from vertex i to vertex j. We denote by $\mathcal{G}^\sigma$ the graph $\mathcal{G}$ with orientation σ. The incidence matrix $D(\mathcal{G}^\sigma)$ of an oriented graph $\mathcal{G}^\sigma$ is the matrix whose rows and columns are indexed by the vertices and edges of $\mathcal{G}$ respectively, such that the i, j entry of $D(\mathcal{G})$ is equal to 1 if the edge j is incoming to vertex i, -1 if edge j is outcoming from vertex i, and 0 otherwise.

The symmetric matrix defined as:

$$L(\mathcal{G}) = \Delta(\mathcal{G}) - A(\mathcal{G}) = D(\mathcal{G}^\sigma)D(\mathcal{G}^\sigma)^T$$

is called the Laplacian of $\mathcal{G}$ and is independent of the choice of orientation σ. It is known that the Laplacian matrix captures many topological properties of the graph. Among those, it is the fact that L is always positive semidefinite and the algebraic multiplicity of its zero eigenvalue is equal to the number of connected components in the graph. The n-dimensional eigenvector associated with the zero eigenvalue is the vector of ones, $\mathbf{1}_n$. If we associate each edge with a positive number and form the diagonal matrix W with rows and columns indexed by the edges of $\mathcal{G}$, then the matrix

$$L_w(\mathcal{G}) = D(\mathcal{G}^\sigma)WD(\mathcal{G}^\sigma)$$

is a weighted Laplacian of $\mathcal{G}$. The weighted Laplacian also enjoys the above properties.

In what follows, we will use graph theoretic terminology to represent the control interconnections between the agents in the group. The connectivity properties of the resulting graph will prove crucial for establishing the stability of the flocking group motion.

IV. Control Law with Fixed Topology

In this section we will refine the acceleration input of (3) into specific expressions for the components a_{r_i} and a_{θ_i}. Then, angular momentum preservation arguments for the group of mobile agents will be used to derive the dynamics of the heading angles. This forms the starting point of the stability analysis that will follow in the next section.

To represent the control interconnections between the agents we use a graph. In this graph, each vertex corresponds to an agent. The edges capture the dependencies of the agent controllers on the position and orientation of other agents. Whenever two vertices are adjacent, this means that the control laws of these agents depend on each other's configuration. Adjacency in the graph will thus induce a neighboring relation between agents. In Part II, this neighboring relation will also be associated with physical adjacency.

Definition IV.1 (Neighboring graph) *The neighboring graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{W}\}$ is a labeled graph consisting of*

- *a set of vertices, $\mathcal{V}$, indexed by the agents in the group,*
- *a set of edges, $\mathcal{E} = \{(v_i, v_j) \mid v_i \sim v_j \text{ for } v_i, v_j \in \mathcal{V}\}$, containing unordered pairs of vertices representing neighboring relations, and*
- *a set of labels, $\mathcal{W}$, indexed by the edges, and a map associating each edge with a label $w \in \mathcal{W}$, equal to the inverse of the squared distance between the agents that correspond to the vertices adjacent to that edge.*

Assumption IV.2 *The neighboring graph, $\mathcal{G}$, is connected.*

Since $\mathcal{G}$ is constant with respect to time, the above assumption ensures that $\mathcal{G}$ will remain connected for all time. By driving the orientation of each agent to the (weighted) average of the headings of its neighbors, the group achieves flocking behavior. Collision avoidance and group cohesion are established using an artificial potential field that is dependent entirely on the relative distances between the agents. For each agent i define an artificial potential function V_i that depends on the distance between i and its neighbors:

$$V_i \triangleq \sum_{j \sim i} V_{ij}(\|r_{ij}\|).$$

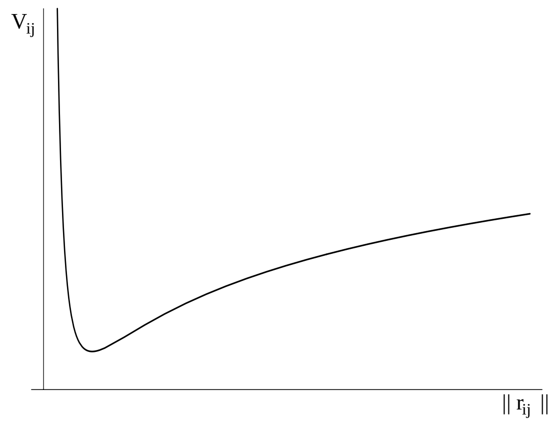

Fig. 2. The artificial potential between two vehicles.

Any potential function $V_{ij}(\|r_{ij}\|)$ can be used, provided that it is symmetric with respect to r_{ij} and r_{ji}. One possible choice could be:

$$V_{ij}(\|r_{ij}\|) = \frac{1}{\|r_{ij}\|^2} + \log \|r_{ij}\|^2 ,$$

giving rise to a potential function that is depicted in Figure 2. The control law a_i is then:

$$a_i = -\nabla_{r_i} V_i + a_{\theta_i}, \tag{4}$$

where a_{θ_i} is given as:

$$a_{\theta_i} = -\sum_{j\sim i} \frac{\theta_i - \theta_j}{\|r_{ij}\|^2}(\hat{z} \times v_i). \tag{5}$$

From preservation of angular momentum for the system of agents we have:

$$\begin{aligned} \sum_{i=1}^{N} r_i \times \dot{v}_i &= \sum_{i=1}^{N} r_i \times (-\nabla V_i + a_{\theta_i}) \\ &= \sum_{i=1}^{N} r_i \times a_{\theta_i} - \sum_{i=1}^{N}\sum_{j\sim i} r_{ij} \times \nabla V_{ij} \\ &= -\sum_{i=1}^{N} r_i \times \left(\sum_{j\sim i} \frac{\theta_i - \theta_j}{\|r_{ij}\|^2}(\hat{z} \times v_i) \right) \end{aligned} \tag{6}$$

since r_{ij} and ∇V_{ij} are collinear. Equation (6) implies that from the angular momentum point of view, the system of interacting agents is equivalent to the one where only the alignment forces are exerted. This fact is to be expected since it is known that "internal forces" (like the potential field forces) in a system of particles do not contribute to the angular momentum of the system. Since these forces are exerted in a direction normal to the agents velocities, we can rewrite (6) using the acceleration component associated with them

$$\sum_{i=1}^{N} r_i \times (\omega_i \times v_i) = -\sum_{i=1}^{N} r_i \times \left(\sum_{j\sim i} \frac{\theta_i - \theta_j}{\|r_{ij}\|^2} (\hat{z} \times v_i) \right) \Leftrightarrow$$

$$\sum_{i=1}^{N} \omega_i (r_i \cdot v_i) = -\sum_{i=1}^{N} \sum_{j\sim i} \frac{\theta_i - \theta_j}{\|r_{ij}\|^2} \hat{z} (r_i \cdot v_i) \Rightarrow$$

$$\sum_{i=1}^{N} (r_i \cdot v_i)(\dot{\theta}_i + \sum_{j\sim i} \frac{\theta_i - \theta_j}{\|r_{ij}\|^2}) = 0$$

Viewing the last expression as a bilinear form in $r_i \cdot v_i$ and $(\dot{\theta}_i, \theta_i)$ and considering its kernel,

$$\dot{\theta}_i = -\sum_{j\sim i} \frac{\theta_i - \theta_j}{\|r_{ij}\|^2} \tag{7}$$

Consider now the following positive semi-definite function:

$$V_t = \frac{1}{2} \sum_{i=1}^{N} (V_i + v_i^T v_i + \theta_i^2) = \frac{1}{2} \sum_{i=1}^{N} (\sum_{j\sim i} V_{ij} + v_i^T v_i + \theta_i^2).$$

Due to V_i being symmetric with respect to r_{ij} and the fact that $r_{ij} = -r_{ji}$,

$$\frac{\partial V_{ij}}{\partial r_{ij}} = \frac{\partial V_{ij}}{\partial r_i} = -\frac{\partial V_{ij}}{\partial r_j}, \tag{8}$$

and therefore it follows:

$$\frac{d}{dt} \sum_{i=1}^{N} \frac{1}{2} V_i = \sum_{i=1}^{N} \nabla_{r_i} V_i \cdot v_i.$$

Let Ω be the set defined as:

$$\Omega \triangleq \{(r_{ij}, v_i, \theta_i) \mid V_t \leq c,\ i,j = 1, \ldots, N\},$$

which is nonempty for a sufficiently large choice of c, and closed by continuity of V_t. It is also bounded, because boundedness of V_t implies boundedness of all V_{ij}, which in turns implies the boundedness of every r_{ij}, since V_{ij} increases monotonically with r_{ij}. Bounds for v_i follow trivially and θ_i is always bounded in $[-\pi, \pi]$.

Proposition IV.3 *Consider the system of N mobile agents with dynamics* (1) *with initial conditions in Ω. When the agents are steered using the control laws* (4)*, then their heading angles converge to some common value and the distances between them are stabilized to a configuration where the group potential energy, V_t attains a minimum.*

Proof: Taking the time derivative of V_t, we have:

$$\begin{aligned}\dot{V}_t &= \sum_{i=1}^N \nabla_{r_i} V_i \cdot v_i + \sum_{i=1}^N v_i^T a_i + \sum_{i=1}^N \theta_i \dot{\theta}_i \\ &\overset{(4)}{=} \sum_{i=1}^N \nabla_{r_i} V_i \cdot v_i + \sum_{i=1}^N v_i^T (-\nabla_{r_i} V_i^T + a_{\theta_i}) + \sum_{i=1}^N \theta_i \dot{\theta}_i \\ &\overset{(5)}{=} \sum_{i=1}^N v_i^T \left(-\sum_{j\sim i} \frac{\theta_i - \theta_j}{\|r_{ij}\|^2} (\hat{z} \times v_i) \right) + \sum_{i=1}^N \theta_i \dot{\theta}_i \\ &= \sum_{i=1}^N \theta_i \dot{\theta}_i. \end{aligned} \tag{9}$$

Expanding (7) we obtain:

$$\dot{\theta}_i = -\sum_{j\sim i} \frac{\theta_i}{\|r_{ij}\|^2} + \sum_{j\sim i} \frac{\theta_j}{\|r_{ij}\|^2}$$

so that (9) can be written as:

$$\dot{V}_t = -\theta^T L_w \theta,$$

where L_w is the weighted Laplacian of the neighboring graph $\mathcal{G}$. The weight on each edge is the inverse squared distance between the agents at the vertices adjacent to the edge. Matrix L_w is positive semidefinite and since $\mathcal{G}$ is assumed connected, the only eigenvector associated with the zero eigenvalue is the N dimensional vector of ones, $\mathbf{1}_N$:

$$\mathbf{1}_N \triangleq (\underbrace{1,\dots,1}_{N})^T.$$

Since $\dot{V}_t \leq 0$, Ω is positively invariant. Applying LaSalle's invariant principle on (1) in Ω we conclude that all trajectories converge to the largest invariant set in $\{(r_{ij}, v_i, \theta_i) \mid \dot{V}_t = 0,\ i,j = 1,\dots,N\}$. Equality $\dot{V}_t = 0$ holds only at configurations where all agents have the same constant heading, $\theta_1 = \cdots = \theta_N = \bar{\theta}$. Let S_θ be the set where all orientations are the same:

$$S_\theta \triangleq \{(r_1, \theta_1, \dots, r_N, \theta_N) \mid \theta_1 = \cdots = \theta_N = \bar{\theta}\}.$$

In this set, we have:

$$\tan\bar{\theta} = k = \frac{\dot{y}_1}{\dot{x}_1} = \cdots \frac{\dot{y}_N}{\dot{x}_N}.$$

Differentiating $\frac{\dot{y}_i}{\dot{x}_i} = k$ we get:

$$\frac{d}{dt}\left(\frac{\dot{y}_i}{\dot{x}_i}\right) = 0 \Rightarrow \frac{\ddot{y}_i}{\ddot{x}_i} = \frac{\dot{y}_i}{\dot{x}_i} = k = \frac{a_{y_i}}{a_{x_i}} = \frac{(\nabla_{r_i} V_i)_y}{(\nabla_{r_i} V_i)_x}.$$

This implies that the potential force on i is aligned with its velocity. We distinguish two cases:

1. Case: $-\nabla_{r_i} V_i \cdot v_i \leq 0$. If we take the function $V_{v_i} = \frac{1}{2} v_i^T v_i$, then we see that $\dot{V}_{v_i} = v_i \dot{v}_i = -\nabla_{r_i} V_i v_i \leq 0$. The dynamics of v_i is now $\dot{v}_i = a_i = -\nabla_{r_i} V_i$, which implies that v_i will converge to the largest invariant set in $S_{v_i} = \{(r_i, v_i) \mid \dot{V}_{v_i} = 0\}$. Since $\nabla_{r_i} V_i$ and v_i are aligned, S_{v_i} contains configurations where either $\nabla_{r_i} V_i = 0$ or $v_i = 0$. The latter configurations are not invariant, unless $\nabla_{r_i} V_i = 0$, because $\dot{v}_i = -\nabla_{r_i} V_i$. Therefore, the system will converge to configurations where $-\nabla_{r_i} V_i = 0$ which correspond to minima of the artificial potential of V_i.
2. Case: $-\nabla_{v_i} V_i \cdot v_i > 0$. Then the time derivative of the artificial potential for i will be $\dot{V}_i = \nabla_{r_i} V_i v_i < 0$, which means that V_i is monotonically decreasing, and eventually reach one of its minima.

In any case therefore, the system converges to a minimum of $\sum_i^N V_i$. ■

Collision avoidance is guaranteed since in all configurations where $r_{ij} = 0$ for some $i, j \in \{1, \ldots, N\}$, the function V_t tends to infinity implying that these configurations lay in the exterior of any Ω with c bounded.

V. Simulations

In this Section we verify numerically the stability results obtained in Section IV. In the simulation example, the group consists of ten mobile agents with identical second order dynamics. Initial conditions (positions and velocities) were generated randomly within a ball of radius $R_0 = 20$[m].

Figures 3-6 and 7-10 give snapshots of the system's evolution, starting from random initial conditions. In the Figures, the position of the agents is represented by small red circles. The yellow circle around each agent marks its sensing neighborhood, which is selected sufficiently large to ensure that the graph is connected at all times. The paths followed by the agents as they move under control laws (4) are shown by green curves.

Figures 3-6 depict the case where the neighboring graph is complete; the motion of each agent is affected by the motion of all other agents. This is a special case where control interaction between the agents is strongest. Figures 7-10 show that flocking motion can still be established for incomplete neighboring graphs. In this scenario, each agent has constant subset of its flockmates as neighbors. The interconnection topology was selected in random and remained fixed throughout the motion.

Simulation verifies that in all cases the system converges to an invariant set, that corresponds to a tight formation and a common heading direction. The shape of the formation which the group converges to, is determined by the artificial potential function.

VI. Conclusions

In this paper we demonstrated how a group of autonomous mobile agents, can be controlled so that it exhibits a cooperative group behavior known as flocking. Flocking requires all the agents to have a common heading and stay close to each other while avoiding collisions. We have modeled flocking and introduced a control scheme that establishes flocking by imposing a fixed topology of interconnections between the agents. The case where this control scheme imposes a dynamic interconnection topology is treated separately in the second part of this paper.

Flocking was shown to be asymptotically stable motion for the closed loop system. The stability analysis was based on LaSalle's invariant principle and was facilitated by known results from algebraic graph theory. The results of this analysis seem to be in agreement with recent findings [11] and suggest that flocking motion can also be established in groups of agents with dynamics, enhanced with collision avoidance capabilities.

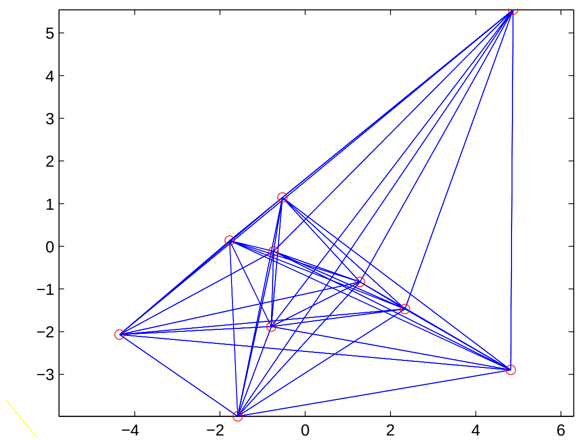

Fig. 3. Complete graph: initial configuration.

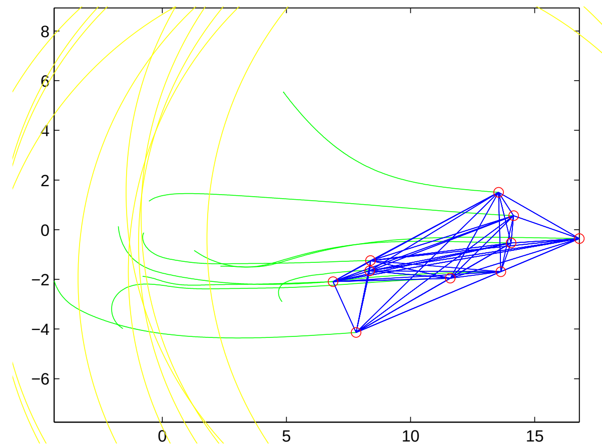

Fig. 4. Complete graph: converging to a common heading.

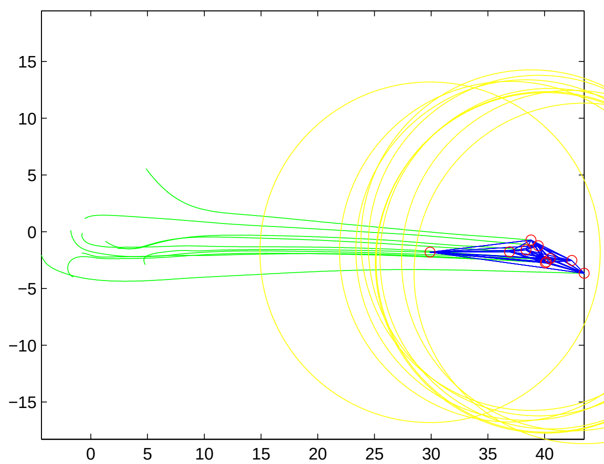

Fig. 5. Complete graph: approaching a potential energy minimum.

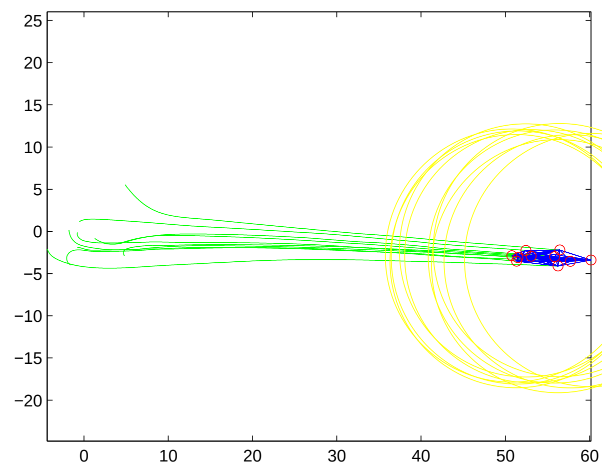

Fig. 6. Complete graph: steady state.

These results seem to be suggesting that it is possible to coordinate multi-agent systems with non-trivial dynamics, using decentralized schemes which combine individual agent properties in non-obvious ways to obtain an aggregated group behavior. The decentralized nature of such control schemes raises hopes for scalable, novel control designs in large scale multi-agent systems.

Acknowledgment

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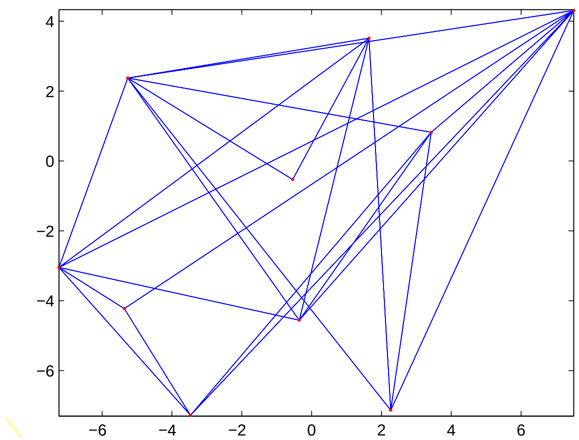

Fig. 7. Incomplete graph: initial configuration.

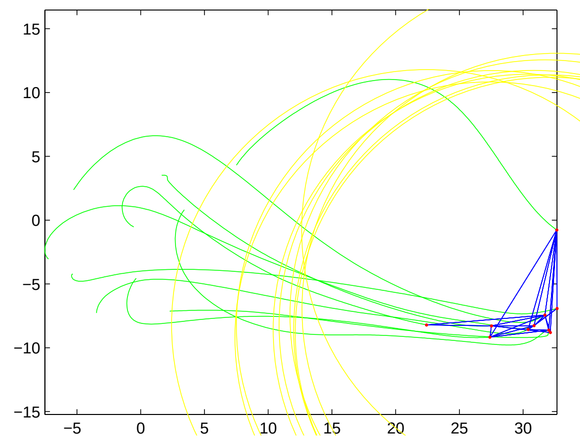

Fig. 8. Incomplete graph: Converging to a common heading.

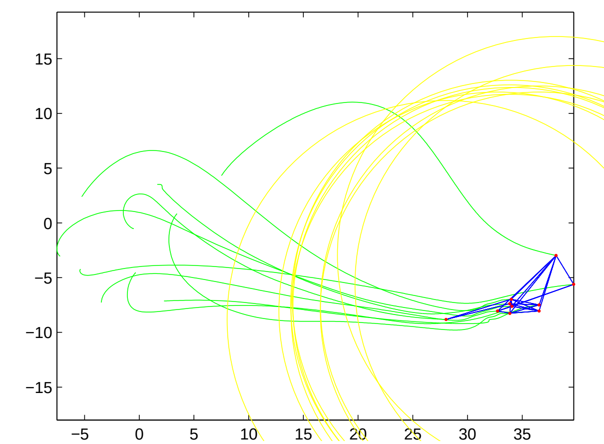

Fig. 9. Incomplete graph: approaching a potential energy minimum.

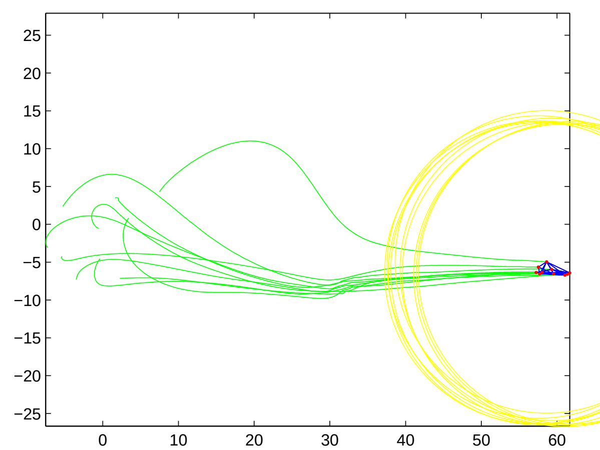

Fig. 10. Incomplete graph: steady state.

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